

# LIMITS ON THE DOMAIN OF COUPLING CONSTANTS FOR BINDING $N$ -BODY SYSTEMS WITH NO BOUND SUBSYSTEMS \* <sup>†</sup>

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## Abstract

We study the domain of coupling constants for which a 3-body or 4-body system is bound while none of its subsystems is bound. Limits on the size of the domain are derived from a variant of the Hall–Post inequalities which relate  $N$ -body to  $(N - 1)$ -body energies at given coupling. Possible applications to halo nuclei and hypernuclei are briefly outlined.

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For the sake of clarity, we shall define in this Letter a “halo” as a 3-body quantum system that is bound while none of its 2-body subsystems has a discrete spectrum. More generally an  $N$ -body halo is bound while none of its subsets is stable against spontaneous dissociation. This is more restrictive than the usual meaning of a weakly bound system with a very extended wave function. These systems are sometimes called “Borromean” [1], after Borromean rings, which are interlaced in such a subtle topological way that if any of them is removed, the other two would be unlocked.

Halo states are seen in nuclear physics [1,2]. For instance the  $(\alpha, n, n)$  system is bound ( ${}^6\text{He}$ ), while  $(\alpha, n)$  and  $(n, n)$  systems are both unbound. There is a cooperative effort of all attractive potentials to achieve the binding of  ${}^6\text{He}$ .

The halo phenomenon shows up in simple potential models, as seen from explicit calculations on specific isotopes [1–3], or from the rich literature on the related Efimov effect [4] or Thomas collapse [5], which prove that a 3-body system is, indeed, more easily bound than a 2-body one.

Consider a short-range potential  $gV(r)$  acting between two particles of mass  $m$  separated by  $r$ . Even if  $V(r)$  is attractive, a minimal strength  $g_2^c/m$  is needed to achieve binding, where  $g_2^c$  is independent of  $m$ . For instance, in a Yukawa potential  $V = -(\exp(-\mu r))/r$ , one can fix the energy and distance scales so that  $m = \mu = 1$  without loss of generality, and one finds  $g_2^c \simeq 1.680$  [6]. (A simple argument by Dyson and Lenard [7] shows that  $g_2^c > \sqrt{2}$ .) Now, if one considers three identical bosons with mass  $m = 1$  interacting through  $\sum(\exp(-r_{ij})/r_{ij})$ , where  $r_{ij}$  is the relative distance between particles  $i$  and  $j$ , one can look at bound states by variational methods or by solving the Faddeev equations, and one finds binding for  $g \geq g_3^c$ , with  $g_3^c/g_2^c \simeq 0.804$ . We call this a 20% window for halo binding.

If one repeats the above calculations for other short-range potentials, one finds similar values for  $g_3^c/g_2^c$ , for instance 0.801 for an exponential form  $V(r) = -\exp(-r)$ , or 0.794 for a Gaussian  $V = -\exp(-r^2)$ . Such quasi-universality, already noticed in [8], is not too surprising. In the weak coupling limit, the wave function extends very far away and does not really probe the detailed structure of the potential. Any short-range interaction is almost equivalent to a delta function in this limit.

It seems therefore very likely that the window for halo phenomena is limited, i.e.,  $g_3^c/g_2^c$  cannot be made arbitrarily small by tuning the shape of  $V(r)$ . We shall show below that

$$\frac{g_3^c}{g_2^c} \geq \frac{2}{3} \quad (1)$$

for any potential, and we outline how to derive similar inequalities for asymmetric 3-body systems, or for 4-body or more complicated systems.

The method of deriving (1) is directly inspired by the Hall–Post inequalities [9], which have been recently rediscovered [10], applied to hadron spectroscopy [11] and generalized to the case of unequal masses [12]. Earlier applications were mostly motivated by considerations on the stability or instability of matter. The Hamiltonian for three bosons of mass  $m$  can be written as

$$\begin{aligned} H_3 &= \sum_{i=1}^3 \frac{\mathbf{p}_i^2}{2m} + g \sum_{i<j} V(r_{ij}) \\ &= \frac{(\sum \mathbf{p}_i)^2}{6m} + \sum_{i<j} \frac{2}{3m} \left( \frac{\mathbf{p}_i - \mathbf{p}_j}{2} \right)^2 + gV(r_{ij}), \end{aligned} \quad (2)$$

i.e., introducing the translation-invariant part  $\tilde{H}_N$  of the Hamiltonian  $H_N$

$$\tilde{H}_3(m, g) = \sum_{i < j} \tilde{H}_2^{(ij)}(3m/2, g) = \frac{2}{3} \sum_{i < j} \tilde{H}_2^{(ij)}(m, 3g/2). \quad (3)$$

Hence, from a simple application of the variational principle, the ground-state energies  $E_N$  for a given (large enough) coupling  $g$  satisfy

$$E_3(m, g) \geq 3E_2(3m/2, g) = 2E_2(m, 3g/2), \quad (4)$$

which is the simplest form of the Hall–Post inequalities. The decomposition (3) also implies that if  $\tilde{H}_3$  has to support a bound state, each  $\tilde{H}_2$  should produce a negative expectation value in the corresponding wave function, and thus  $3g/2 \geq g_2^c$ , q.e.d.

A straightforward generalization of (3) to  $N$  bosons is [10]

$$\tilde{H}_N(m, g) = \frac{1}{N-2} \sum_{k=1}^N \tilde{H}_{N-1}^{[k]} \left( \frac{Nm}{N-1}, g \right), \quad (5)$$

where the superscript in  $\tilde{H}_{N-1}^{[k]}$  means that the  $k$ -th particle is omitted. Saturating with the ground state of  $\tilde{H}_N$  gives

$$g_N^c \geq \frac{N-1}{N} g_{N-1}^c, \quad (6)$$

i.e.  $Ng_N^c$  increases with  $N$ .

For numerical illustration in the  $N = 4$  case, we adopted a variational method that is widely used in quantum chemistry [13]. It is based on trial wave functions of the type

$$\Psi(\mathbf{x}_i) = \sum_n c^{(n)} \exp \left[ -\frac{1}{2} \sum_{i,j} a_{ij}^{(n)} \mathbf{x}_i \cdot \mathbf{x}_j \right], \quad (7)$$

where  $\{\mathbf{x}_i\}$  is a set of relative Jacobi coordinates. Symmetry is properly implemented by imposing relations between neighbouring coefficients  $c^{(n)}$  and definite-positive matrices  $a^{(n)}$ . After numerical minimization, we obtain  $g_4^c/g_2^c \simeq 0.67$  for a Yukawa potential, i.e. a 13% window for a genuine 4-boson halo, once  $g_4^c/g_2^c$  is subtracted from  $g_3^c/g_2^c$ .

Note that the bounds (1) and (6) are not expected to be saturated, since the Hall–Post inequalities become equalities only for harmonic oscillators, which are far from the short-range potentials we consider here.

Similar inequalities can be written down in a variety of situations. Let us give some examples. Consider first  $N$  identical particles with mass  $m = 1$  in the field of an infinitely massive source. The Hamiltonian is defined as

$$H_N = \sum_{i=1}^N \left[ \frac{\mathbf{p}_i^2}{2m} + hV(r_i) \right] + g \sum_{i < j} W(r_{ij}). \quad (8)$$

The short-range attractive potentials  $V$  and  $W$  can be normalized such that a single particle is trapped around the source for  $h > 1$ , and two particles are bound together for  $g > 1$ . The non-trivial domain is thus to be found inside the square ( $h < 1, g < 1$ ).

Let  $h_N(g)$  be the boundary for binding  $N$  particles around the source. This means that halo type of binding occurs between  $h_N(g)$  and  $h_{N-1}(g)$ . One expects all curves to merge at  $A(h = 1, g = 0)$  as  $g \rightarrow 0$ , since the particles become independent in this limit. If one takes the expectation value of the identity

$$H_3(h, g) = \frac{1}{2} \sum_{i < j} H_2^{(i,j)}(h, 2g) \quad (9)$$

within the ground state of  $H_3$ , one gets

$$h_3(g) \geq h_2(2g), \quad (10)$$

and more generally

$$h_N(g) > h_{N-1} \left( \frac{N-1}{N-2} g \right). \quad (11)$$

For  $N = 2$ , one can write the simple decomposition

$$H_2 = \left[ \frac{\alpha}{2} \mathbf{p}_1^2 + hV(r_1) \right] + \left[ \frac{\alpha}{2} \mathbf{p}_2^2 + hV(r_2) \right] + \left[ \frac{1-\alpha}{2} (\mathbf{p}_1^2 + \mathbf{p}_2^2) + gW(r_{12}) \right], \quad (12)$$

for any  $0 \leq \alpha \leq 1$ . To get  $\langle H_2 \rangle < 0$ , one needs at least one of the square brackets having a negative expectation value. This excludes the triangle  $\{h \leq \alpha, g \leq (1-\alpha)\}$  shown in Fig. 1.

Two remarks on this simple lower bound are in order. First, the actual boundary is expected to be concave. The couplings  $h$  and  $g$  enter the Hamiltonian linearly, so if the minimum  $E$  of  $H_2$  vanishes at both  $P(h, g)$  and  $P'$ , one has [14]

$$E(\lambda P + (1-\lambda)P') \geq \lambda E(P) + (1-\lambda)E(P') = 0, \quad (13)$$

for any  $0 \leq \lambda \leq 1$ .

Secondly, while the limit  $A(h = 1, g = 0)$  of truly independent particles obviously belongs to the boundary,  $B(h = 0, g = 1)$  is likely to be in the continuum, since a weakly bound (1,2) system needs a minimal attraction  $h$  to remain trapped by the source.

A more elaborate decomposition leads to an improved boundary which better complies with the above remarks. We provisionally restore a finite mass  $M$  for the third particle and write as in [12]

$$\begin{aligned} H(M) &= \frac{\mathbf{p}_1^2}{2} + \frac{\mathbf{p}_2^2}{2} + \frac{\mathbf{p}_3^2}{2M} + hV(r_{13}) + hV(r_{23}) + gW(r_{12}) \\ &= (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \cdot (b\mathbf{p}_1 + b'\mathbf{p}_2 + b'\mathbf{p}_3) \\ &\quad + \sum_{i=1,2} \left( \frac{1+x}{1+2x} \right)^2 \left( \frac{1}{2} + \frac{1}{M} \right) \left( \frac{\mathbf{p}_i - x\mathbf{p}_3}{1+x} \right)^2 + hV(r_{i3}) \\ &\quad + \frac{4}{(1+2x)^2} \left( x(1+x) - \frac{1}{2M} \right) \left( \frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right)^2 + gW(r_{12}), \end{aligned} \quad (14)$$

where the momentum  $(\mathbf{p}_i - x\mathbf{p}_3)/(1+x)$  is the conjugate of the relative distance  $\mathbf{r}_{i3}$ , and  $b$  and  $b'$  are known functions of  $M$  and  $x$ . In the limit  $M \rightarrow \infty$ , we read off from (14) that  $H_2$  would never support a bound state if

$$h \left( \frac{1+2x}{1+x} \right)^2 < 1 \quad \text{and} \quad g \frac{(1+2x)^2}{4x(1+x)} < 1. \quad (15)$$

This is the inner part of the parabola shown in Fig. 1, from which we get a crude lower limit on the minimal coupling  $h_3(g)$  to bind three bosons around the source, as per Eq. (10).

The decomposition (14) can be used for finite  $M$ . Consider for instance the case where  $M = 1$  and  $g = 0$ . One should restrict to  $x(1+x) > 1/2$  in order not to introduce a negative reduced mass. The two particles, which do not interact with each other, can be bound simultaneously below the critical coupling  $h = 1$  for individual coupling. Each particle benefits from the increase of the reduced mass provided by the other. However, by choosing the optimal parameter  $x$  in (14), one can easily deduce that the window for halo is limited to  $h > 1/2 + \sqrt{3}/4 \simeq 0.93$ , i.e. at most 7%.

In a situation where the three masses or the three couplings are different, the most general decomposition involves two parameters [12], instead of the single  $x$  in (14). So the analysis becomes slightly more involved.

We now write the Hamiltonian for two identical particles of mass  $m$ , and two others of mass  $M$

$$H_4 = \sum_{i=1,2} \frac{\mathbf{p}_i^2}{2m} + \sum_{j=3,4} \frac{\mathbf{p}_j^2}{2M} + g_{12}V_{12}(r_{12}) + g_{34}V_{34}(r_{34}) + g_{mM} \sum_{i,j} V_{mM}(r_{ij}), \quad (16)$$

including up to three different potentials. It can be rewritten as

$$\begin{aligned} H_4 = & (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \cdot (b\mathbf{p}_1 + b\mathbf{p}_2 + b'\mathbf{p}_3 + b'\mathbf{p}_4) \\ & + a_{12} \left( \frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right)^2 + g_{12}V_{12}(r_{12}) + a_{34} \left( \frac{\mathbf{p}_3 - \mathbf{p}_4}{2} \right)^2 + g_{34}V_{34}(r_{34}) \\ & + \sum_{i,j} \bar{a} [\alpha \mathbf{p}_i - (1-\alpha)\mathbf{p}_j]^2 + g_{mM}V_{mM}(r_{ij}) \end{aligned} \quad (17)$$

For given  $\alpha$ , one can solve for  $b$  and  $b'$ , as well as for the inverse masses, with the result

$$\begin{aligned} a_{12} &= \frac{1}{m}(1-\alpha^2) - \frac{1}{M}\alpha^2, \\ a_{34} &= -\frac{1}{m}(1-\alpha)^2 + \frac{1}{M}\alpha(2-\alpha), \\ \bar{a} &= \frac{1}{4m} + \frac{1}{4M}. \end{aligned} \quad (18)$$

If one rescales the couplings to the critical value for 2-body binding with the appropriate inverse reduced mass,  $m^{-1}$  for  $g_{12}$ ,  $M^{-1}$  for  $g_{34}$ , and  $(m^{-1} + M^{-1})/2$  for  $g_{mM}$ , then  $H_4$  cannot support a bound state if simultaneously

$$\begin{aligned} g_{12} &\leq 1 - \alpha^2 - (m/M)\alpha^2, \\ g_{34} &\leq -(M/m)(1-\alpha)^2 + \alpha(2-\alpha), \\ g_{mM} &\leq 1/2. \end{aligned} \quad (19)$$

Interestingly, the condition on  $g_{mM}$  decouples. Hence the domain for 4-body binding without 2-body binding consists at most of  $1/2 < g_{mM} < 1$ , and, in the  $(g_{12}, g_{34})$  plane, the area

between the unit square and a parabola, as shown in Fig. 2. One should of course exclude the domain corresponding to 3-body binding to get a genuine halo.

Our interest in 3-body systems was clearly triggered by nuclei like  ${}^6\text{He}$  or  ${}^{11}\text{Li}$  with two neutrons weakly attached to a compact nucleus [1]. We assume a spin singlet state for the two neutrons, so that their spatial wave function is symmetric. States with more than two neutrons in the halo unfortunately escape direct application of our results, because of the Pauli principle. Our result on  $(m, m, M, M)$  configurations is perhaps relevant for  $(n, n, \Lambda, \Lambda)$  hypernuclei with strangeness  $S = -2$ , a field of intense theoretical studies [15].

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# FIGURES

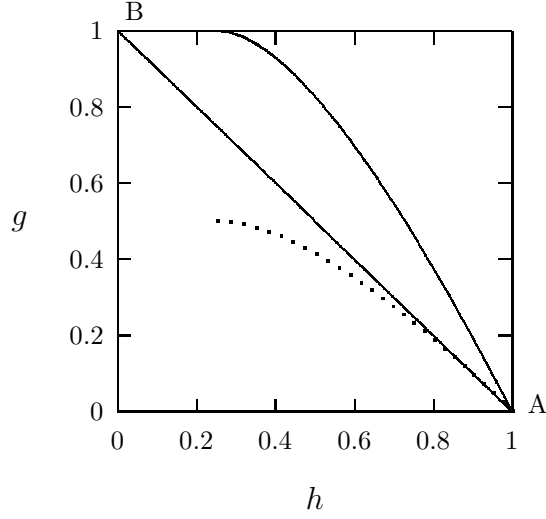


FIG. 1. Lower limit for the domain of halo binding of two bosons in the field of a static source. The coupling to the source is denoted  $h$ , while  $g$  is the interparticle coupling. Two-body binding occurs for  $h > 1$  or  $g > 1$ . The straight line AB is deduced from the simple decomposition (12), the parabola from (14). The dotted parabola is a lower limit for binding three bosons in this central field, as deduced from (9).

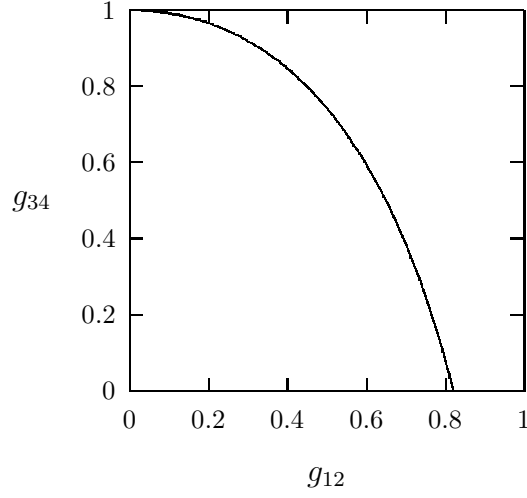


FIG. 2. Lower limit for binding four particles with masses  $(m, m, M, M)$ . The couplings  $g_{12}$  and  $g_{34}$  are normalized to the critical coupling for binding  $(m, m)$  and  $(M, M)$ , respectively. Meanwhile the  $(m, M)$  coupling should be at least half the critical coupling for binding  $m$  to  $M$ . A value  $M/m = 2$  is assumed here for the drawing.